

# $\mathcal{N} = 2$ Heterotic-Type II duality and bundle moduli

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**ABSTRACT:** Heterotic string compactifications on a  $K3$  surface  $\mathfrak{S}$  depend on a choice of hyperkähler metric, anti-self-dual gauge connection and Kalb-Ramond flux, parametrized by hypermultiplet scalars. The metric on hypermultiplet moduli space is in principle computable within the  $(0, 2)$  superconformal field theory on the heterotic string worldsheet, although little is known about it in practice. Using duality with type II strings compactified on a Calabi-Yau threefold, we predict the form of the quaternion-Kähler metric on hypermultiplet moduli space when  $\mathfrak{S}$  is elliptically fibered, in the limit of a large fiber and even larger base. The result is in general agreement with expectations from Kaluza-Klein reduction, in particular the metric has a two-stage fibration structure, where the  $B$ -field moduli are fibered over bundle and metric moduli, while bundle moduli are themselves fibered over metric moduli. A more precise match must await a detailed analysis of  $R^2$ -corrected ten-dimensional supergravity.

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## 1 Introduction

Nearly thirty years after their inception [1, 2], heterotic strings compactified on Calabi-Yau threefolds continue to be a framework of choice for constructing MSSM or GUT-like perturbative string vacua. However, despite this phenomenological appeal, a complete description of the low-energy effective theory of such compactifications is still missing so far. This is due in part to the difficulty in understanding the gauge bundle moduli space, but also to the subtle nature of the  $B$ -field in heterotic string theory [3, 4]. Indeed, anomaly cancellation requires that  $B$  transforms non-trivially under diffeomorphisms and gauge transformations, so that the gauge invariant field strength (with  $\alpha' = 1$ )

$$H = dB + \frac{1}{4}(\omega_G - \omega_L) \quad (1.1)$$

involves a contribution from the Chern-Simons forms  $\omega_G$  and  $\omega_L$ , for the gauge group  $G$  ( $E_8 \times E_8$  or  $SO(32)$ ) and the Lorentz group  $SO(1, 9)$ , respectively. The equation of motion and Bianchi identity

$$d \star_g H = 0, \quad dH = \frac{1}{4}(\text{Tr} F \wedge F - \text{Tr} R \wedge R), \quad (1.2)$$

imply that  $B$  cannot be expanded on a basis of harmonic forms, except in the special case of the standard embedding, where the r.h.s. of the Bianchi identity vanishes point wise. Furthermore, supersymmetry requires additional higher-derivative couplings in the  $D = 10$  supergravity Lagrangian [5], which greatly complicate the Kaluza-Klein reduction.

Even in the simpler case of heterotic strings compactified on  $K3$ , preserving  $\mathcal{N} = 1$  supersymmetries in 6 dimensions (or equivalently, for heterotic strings on  $K3 \times T^2$ , preserving

$\mathcal{N} = 2$  supersymmetries in 4 dimensions), the knowledge of the hypermultiplet moduli space  $\mathcal{M}_H$  describing metrics  $g$ , gauge bundles  $F$  and  $B$ -fields on  $K3$  is incomplete. In principle it is entirely determined within the  $(0, 4)$  worldsheet SCFT at tree-level, by the usual decoupling argument between hypermultiplets and vector multiplets, which include the heterotic dilaton together with the metric and gauge bundle moduli on  $T^2$  [6]. The metric on  $\mathcal{M}_H$  has however remained largely unknown, aside from the ‘standard embedding’ locus where the SCFT has enhanced  $(4, 4)$  supersymmetry. This ignorance, along with an incomplete understanding of non-perturbative effects in type II Calabi-Yau vacua, has prevented detailed tests of heterotic/type II duality [7, 8] in the hypermultiplet sector, beyond the early attempts in [9–11]. The situation on the type II side has considerably improved in recent years (see [12, 13] for recent reviews), and it is therefore natural to strive for a similar improvement on the heterotic side. Although  $\mathcal{N} = 2$  vacua are not phenomenologically relevant, the lessons learned in this process will likely be useful for heterotic compactifications on Calabi-Yau threefolds as well [14].

The Kaluza-Klein reduction of the ten-dimensional heterotic supergravity on  $K3$  including bundle and  $B$ -field moduli was discussed recently in [15], building on earlier work [16, 17]. While the contribution of the gauge Chern-Simons form  $\omega_G$  to the equation of motion was included, the reduction did not include the Lorentz Chern-Simons form  $\omega_L$ , nor the  $R^2$  couplings related to it by supersymmetry. As a result, the low-energy effective action was not supersymmetric, in particular the metric on hypermultiplet moduli space was not quaternion-Kähler (QK). One of the general lessons from [15], however, was that, unlike what has been often stated or implicitly assumed in the literature, the hypermultiplet moduli space  $\mathcal{M}_H$  is definitely *not* a bundle of hyperkähler (HK) spaces  $\mathcal{M}_F(g, V)$ , parametrizing the bundle, over the moduli space of the  $(4, 4)$  SCFT, corresponding to the HK metric  $g$  on  $K3$  along with the  $B$ -field moduli. Rather, it must have a two-stage fibration structure

$$\begin{array}{ccc} \mathcal{M}_B(g, F) & \rightarrow & \mathcal{M}_H \\ & \downarrow & \\ \mathcal{M}_F(g) & \rightarrow & \mathcal{M}_{g,F} \\ & \downarrow & \\ & & \mathcal{M}_g \end{array} \tag{1.3}$$

where the  $B$ -field moduli  $\mathcal{M}_B(g, F)$  live in a torus bundle, fibered over both the metric moduli  $\mathcal{M}_g$  and the bundle moduli  $\mathcal{M}_F(g)$ , the latter being fibered over  $\mathcal{M}_g$  as well. As we discuss in §2, this structure is an immediate consequence of the Bianchi identity (1.2), as recognized early on in [4]. The main goal of this paper is to investigate the structure of this two-stage fibration and to understand how it can be compatible with the quaternion-Kähler (QK) property of the total hypermultiplet moduli space  $\mathcal{M}_H$ , which is a necessary requirement for supersymmetry [18].

For this purpose, our strategy will be to assume that heterotic/type II duality holds, use our knowledge of the hypermultiplet moduli space on the type IIB side in a suitable limit, and find a change of coordinates (or duality map) which displays the two-stage fibration structure

(1.3). In order for heterotic/type II duality to apply, we assume that the  $K3$  manifold  $\mathfrak{S}$  on the heterotic side is elliptically fibered, while the Calabi-Yau three-fold on the type IIB side is  $K3$ -fibered [19]. Furthermore, we are interested in the limit where one can ignore  $g_s$ -corrections on the type IIB side and  $\alpha'$ -corrections on the heterotic side. To this end, we assume that the type IIB ten-dimensional string coupling is weak (so that, on the heterotic side, the area of the elliptic fiber is much smaller than the base), and that the volume of  $\mathfrak{S}$  is large in heterotic string units (so that, on the type IIB side, the area of the base of the  $K3$  fibration is also large in type II string units). In this regime, and in the special case of heterotic compactifications with a rigid bundle, a suitable duality map was constructed in [19, 20], which identifies the hypermultiplet moduli space on the type IIB side (parametrizing the string coupling, Neveu-Schwarz axion, Kähler structures and associated RR-potentials) with the heterotic hypermultiplet moduli space.

In this paper, we extend this duality map to the case of heterotic  $K3$  compactifications with non-rigid bundles. As in [19], we identify the heterotic bundle moduli with the Kähler moduli associated to reducible singular fibers on the type IIB side, along with the corresponding RR moduli.<sup>1</sup> Retaining the first subleading correction to the holomorphic prepotential in the limit where the area of the base of the  $K3$  fibration is large, see (3.3), we find that the type IIB hypermultiplet space, translated into heterotic variables, neatly displays the double fibration structure (1.3). In particular, the HK metric on the bundle moduli space  $\mathcal{M}_F(g)$  is obtained by the rigid  $c$ -map construction from the prepotential  $f(t^i, t^\alpha)$  appearing as the subleading correction in (3.3). This is partially expected since for elliptic  $K3$ 's the bundle moduli space has the structure of a complex integrable system [22], with a semi-flat metric in the limit where the area of the elliptic fiber is much smaller than the base. It is however worth noting that the metric on  $\mathcal{M}_F(g)$  could have been in the more general class of hyperkähler metrics on cotangent bundles of Kähler manifolds constructed in [23]. Another feature of the metric on  $\mathcal{M}_H$  which can be extracted from the dual type II description is the topology of the torus fiber  $\mathcal{M}_B(g, F)$  over bundle moduli space. We find that the curvature of the Levi-Civita connection on this torus bundle is in nice agreement with the Bianchi identity (1.2). In addition, the classical hypermultiplet metric predicts a precise fibration structure over metric moduli and various volume-suppressed corrections to the moduli space metric, which are necessary for the quaternion-Kähler property. We leave it as an open problem to derive these corrections by reducing 10D heterotic supergravity (including the higher order derivative corrections required by supersymmetry [5]) on  $\mathfrak{S}$ .

The outline of this work is as follows. In §2, we discuss the general structure of the hypermultiplet moduli space in heterotic strings compactified on elliptic  $K3$  with a non-rigid bundle. In §3, we recall basic aspects of heterotic/type II duality and translate the classical type IIB hypermultiplet metric in heterotic variables. We read off the two-stage fibration (1.3) and compare with expectations from the Kaluza-Klein reduction of tree-level supergravity on the heterotic side. We conclude in §4 with a discussion of open issues.

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<sup>1</sup>This identification is analogous to the standard identification between bundle moduli on  $T^2$  and complex structure moduli associated to singular  $K3$  fibers of the CY 3-fold on the type IIA side [11, 21].

## 2 Generalities on heterotic moduli spaces

In this section, we discuss qualitative aspects of the hypermultiplet moduli space in compactifications of the heterotic string on a  $K3$  surface  $\mathfrak{S}$ . The same hypermultiplet moduli space appears in compactifications on  $K3 \times T^2$  down to 4 dimensions, as the additional scalar fields coming from the metric and gauge bundle on  $T^2$  all lie in vector multiplets. For concreteness, we restrict our analysis to elliptically fibered  $K3$  surfaces and the gauge group  $G = E_8 \times E_8$ , so that heterotic/type II duality applies, but most of the considerations below hold more generally. Our notations follow [20].

As mentioned in the introduction, vacua with unbroken  $\mathcal{N} = 2$  supersymmetry are characterized by a hyperkähler metric  $g$  on  $\mathfrak{S}$ , a bundle  $F$  on  $\mathfrak{S}$  with second Chern class  $c_2(F) = \chi(\mathfrak{S}) = 24$  (as follows from the Bianchi identity (1.2)) equipped with an anti-self dual connection  $A$  such that<sup>2</sup>  $F = dA + A \wedge A$ , and a two-form  $B$  on  $\mathfrak{S}$  satisfying the equation of motion (1.2). The metric on the resulting moduli space turns out to be given by a sum of three terms corresponding to the three types of the moduli (see (2.34) below). In the following we discuss each of these contributions separately.

### 2.1 Metric moduli

For what concerns the metric degrees of freedom, it is well known that the moduli space of smooth HK metrics on  $\mathfrak{S}$  is (up to global identifications) the homogeneous space

$$\mathcal{M}_g = \mathbb{R}_\rho^+ \times \left[ \frac{SO(3, n-1)}{SO(3) \times SO(n-1)} \right]_{\gamma_I^x} \quad (2.1)$$

with  $n = 20$  (see e.g. [24]). The first factor corresponds to the volume  $\mathcal{V} = e^{-\rho}$  in heterotic string units, whereas the second factor is parametrized by periods of the triplet of HK forms  $\mathcal{J}^x$ ,  $x = 1, 2, 3$ , along a basis  $\tau_I$  ( $I = 1, \dots, n+2$ ) of  $H_2(\mathfrak{S}, \mathbb{Z})$ ,

$$\gamma_I^x = e^{\rho/2} \int_{\tau_I} \mathcal{J}^x, \quad \eta^{IJ} \gamma_I^x \gamma_J^y = 2\delta^{xy}, \quad (2.2)$$

where  $\eta^{IJ}$  is the inverse of the intersection matrix on  $H_2(\mathfrak{S}, \mathbb{Z})$ . The periods  $\gamma_I^x$  may be organized into a  $SO(3, n-1)$  symmetric matrix

$$M_{IJ} = \int_{\mathfrak{S}} \omega_I \wedge \star \omega_J = -\eta_{IJ} + \gamma_I^x \gamma_J^x, \quad (2.3)$$

where  $\omega_I$  is a 2-form dual to  $\tau_I$ , satisfying the following property

$$(M^{-1})^{IJ} = \eta^{IK} \eta^{JL} M_{KL} \equiv M^{IJ}. \quad (2.4)$$

The matrix  $M_{IJ}$  parametrizes the conformal class of the HK metric on  $\mathfrak{S}$ . In terms of this matrix and the volume coordinate  $\rho$ , the  $SO(3, n-1)$  invariant metric on  $\mathcal{M}_g$  reads

$$ds_g^2 = \frac{1}{2} d\rho^2 - \frac{1}{4} dM_{IJ} dM^{IJ}. \quad (2.5)$$

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<sup>2</sup>We abuse notation and denote by the same letter  $F$  the bundle and its curvature.

For singular  $K3$ 's with  $k$  shrinking cycles the moduli space has the same structure as above, with  $\tau_I$  running over a basis of  $n + 2 = 22 - k$  unobstructed cycles.

For elliptically fibered  $K3$ , the description given above can be further refined. Let us denote by  $\mathcal{B}$  the base of the elliptic fibration,  $\mathcal{E}$  the elliptic fiber, and  $\tau_A$  a basis of the transcendental lattice on  $\mathfrak{S}$ . Here the index  $A$  is running over a set of cardinality  $n$ , and we choose the basis of  $H_2(\mathfrak{S}, \mathbb{Z})$  to be given by  $\tau_I = (\mathcal{B} + \mathcal{E}, \mathcal{E}, \tau_A)$  with  $\{I\} = \{1, 2, A\}$ . Furthermore, we can fix the  $SO(3)$  rotation symmetry among the three complex structures by choosing a complex structure adapted to the elliptic fibration, i.e.  $\int_{\mathcal{E}} \mathcal{J} = 0$  where  $\mathcal{J} = \mathcal{J}^1 + i\mathcal{J}^2$  is the holomorphic 2-form in this complex structure. In this setup the intersection form  $\eta_{IJ}$  is given by

$$\eta_{IJ} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \eta_{AB} \end{pmatrix}, \quad (2.6)$$

where  $\eta_{AB}$  is the intersection form on the transcendental lattice, and the periods  $\gamma_I^x$  can be parametrized as

$$\begin{aligned} \gamma_1 &= -\eta^{AB} \gamma_A v_B, & \gamma_2 &= 0, & \gamma_A &= \frac{X_A}{\sqrt{\eta^{BC} X_B X_C}}, \\ \gamma_1^3 &= e^{-R/2} - \frac{v^2}{2} e^{R/2}, & \gamma_2^3 &= e^{R/2}, & \gamma_A^3 &= e^{R/2} v_A \end{aligned} \quad (2.7)$$

with  $\gamma_I = \frac{1}{2}(\gamma_I^1 + i\gamma_I^2)$ . The complex structure moduli  $X^A$  are subject to the constraint  $X^A \eta_{AB} X^B = 0$  following from the orthogonality relation (2.2), while the  $v_A$ 's are the (real) Kähler moduli. The complex structure moduli are conveniently organized into a  $SO(2, n-2)$  symmetric matrix

$$M_{AB} = -\eta_{AB} + 2X_{AB}, \quad X_{AB} = \frac{X_A \bar{X}_B + \bar{X}_A X_B}{\eta^{CD} X_C \bar{X}_D}, \quad (2.8)$$

which allows to rewrite the metric (2.5) on  $\mathcal{M}_g$  as follows

$$ds_g^2 = \frac{1}{2} d\rho^2 + \frac{1}{2} dR^2 - \frac{1}{4} dM_{AB} dM^{AB} + e^R M_{AB} dv^A dv^B, \quad (2.9)$$

where the third term may be equivalently written as

$$-\frac{1}{4} dM_{AB} dM^{AB} = 4 \frac{dX^A d\bar{X}^B}{\eta_{CD} X^C \bar{X}^D} (X_{AB} - \eta_{AB}). \quad (2.10)$$

This implies that the second factor in (2.1) has the following bundle structure

$$\begin{aligned} \mathbb{R}_{v^A}^{2, n-2} &\longrightarrow \left[ \frac{SO(3, n-1)}{SO(3) \times SO(n-1)} \right]_{\gamma_I^x} \\ &\quad \downarrow \\ \mathbb{R}_R^+ &\times \left[ \frac{SO(2, n-2)}{SO(2) \times SO(n-2)} \right]_{X^A}, \end{aligned} \quad (2.11)$$

where the fiber parametrizes the Kähler structure while the base parametrizes the complex structure compatible with the elliptic fibration.

## 2.2 Bundle moduli

We now turn to the gauge bundle moduli. For simplicity, we restrict to bundles whose structure group lies in a subgroup  $H = SU(N) \subset G$ . This breaks the gauge symmetry from  $G$  down to  $G_0$ , where  $G_0$  is the commutant<sup>3</sup> of  $H$  inside  $G$ . The Bianchi identity (1.2) fixes the second Chern class to be  $c_2(F) = 24$ , while  $c_1(F) = 0$  for  $SU(N)$  bundles. Supersymmetry requires the connection  $A$  to be anti-self-dual,  $\star F = -F$ .

For fixed metric  $g$  on  $\mathfrak{S}$  and bundle topology, the space of gauge inequivalent anti-self-dual connections is a finite-dimensional hyperkähler space  $\mathcal{M}_F(g)$  of quaternionic dimension [27]

$$m = c_2(F) h(H) - \dim(H), \quad (2.12)$$

where  $h(H)$  is the dual Coxeter number of  $H$ , equal to  $N$  for  $H = SU(N)$ . The  $S^2$  family of complex structures on  $\mathcal{M}_F(g)$  arises from the fact that the anti-self dual condition is equivalent to the hermitian-Yang-Mills equations in any complex structure  $J$  on  $\mathfrak{S}$  [28],

$$F^{2,0} = 0, \quad F \wedge \mathcal{J}^3 = 0. \quad (2.13)$$

Solutions to (2.13) are in one-to-one correspondence with semi-stable holomorphic vector bundles [29], thereby providing a complex structure on  $\mathcal{M}_F(g)$ . The tangent space over a given connection  $A_0$  is generated by hermitian one-forms  $\hat{a} \in \Omega^1(\text{End}^h F)$  such that

$$d_{A_0} \hat{a} \in \Omega^{1,1}(\text{End} F), \quad \int \mathcal{J}^3 \wedge d_{A_0} \hat{a} = 0, \quad d_{A_0}^\dagger \hat{a} = 0, \quad (2.14)$$

where  $d_{A_0} \hat{a} = d\hat{a} + [A_0, \hat{a}]$  and the last condition fixes the gauge. By using the complex structure, we can split the connection  $A_0$  and its variation  $\hat{a}$  into their  $(0,1)$  and  $(1,0)$  parts,

$$A_0 = \mathcal{A}_0 + \mathcal{A}_0^\dagger, \quad \hat{a} = a + a^\dagger, \quad \text{where } \mathcal{A}_0, a \in \Omega^{0,1}(\text{End} F). \quad (2.15)$$

so that the tangent space over  $A_0$  is isomorphic to the space of  $\bar{\partial}_{A_0}$ -harmonic  $(0,1)$ -forms with values in  $\text{End} F$ , i.e. the cohomology group  $H^{0,1}(\text{End} F)$ . Hence, its complex dimension  $2m \equiv h^{0,1}(\text{End} F)$  is equal to minus the index

$$\chi(\text{End} F) = \int_{\mathfrak{S}} \text{Td}(\mathfrak{S}) \wedge \text{ch}(\text{End} F) = 2 \text{rk}(\text{End} F) + \text{ch}_2(\text{End} F), \quad (2.16)$$

where  $\text{Td}(\mathfrak{S}) = 1 + \frac{1}{12}c_2(\mathfrak{S})$  and  $\text{ch} = \text{rk} + c_1 + \frac{1}{2}c_1^2 - c_2$ . For an  $SU(N)$  bundle,  $\text{rk}(\text{End} F) = N^2 - 1$ ,  $c_1(\text{End} F) = 0$ ,  $c_2(\text{End} F) = 2Nc_2(F)$ , in agreement with (2.12). It is important to note that the metric on  $\mathcal{M}_F(g)$  depends on the metric  $g$  on  $\mathfrak{S}$  via the connection  $A_0$ , which is assumed to be anti-self-dual with respect to  $g$ .

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<sup>3</sup>At special points in moduli space, e.g. when the bundle becomes point-like, the gauge symmetry may enhance and some additional charged matter may appear. This symmetry enhancement is most easily understood in  $D = 4$ , where it corresponds to standard unHiggsing. Going to the Coulomb branch of this gauge theory typically leads to a new family of vacua with different bundle topology. In six dimensions the transition is more exotic and involves tensionless strings [25, 26]. In this paper we stay away from such transitions.



We denote by  $\xi^k$  ( $k = 1, \dots, 2m$ ) a system of complex coordinates on  $\mathcal{M}_F(g)$ , and choose a family of solutions  $\mathcal{A}(y, \xi^k, g^i)$  of the anti-self-duality equations which intersects each gauge orbit once (here  $y$  denotes a set of coordinates on  $\mathfrak{S}$ , and  $g^i$  denote the metric moduli discussed in §2.1). Of course,  $\mathcal{A}(y, \xi^k, g^i)$  is ambiguous modulo gauge transformations  $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}\mathcal{U}^{-1} - \bar{\partial}\mathcal{U}\mathcal{U}^{-1}$ . In order to cancel this ambiguity, one introduces an  $\text{End}F$ -valued connection  $\Lambda = \Lambda_k d\xi^k$  on  $\mathcal{M}_F(g)$  transforming as  $\Lambda_k \rightarrow \mathcal{U}\Lambda_k\mathcal{U}^{-1} - \partial_{\xi^k}\mathcal{U}\mathcal{U}^{-1}$  in such a way that  $D_{\xi^k}\mathcal{A} \equiv \partial_{\xi^k}\mathcal{A} - \bar{\partial}\Lambda_k + [\mathcal{A}, \Lambda_k]$  transforms linearly,  $D_{\xi^k}\mathcal{A} \rightarrow \mathcal{U}D_{\xi^k}\mathcal{A}\mathcal{U}^{-1}$  [14]. By construction,  $D_{\xi^k}\mathcal{A}$  furnish a basis for  $H^{0,1}(\text{End}F)$ . Similarly, we need covariant derivatives  $D_{g^i}\mathcal{A}$  with respect to metric moduli, which cancel the gauge dependence of ordinary derivatives  $\partial_{g^i}\mathcal{A}$  (note that  $D_{g^i}\mathcal{A}$  are in general *not* (0,1) forms, since a variation of  $g^i$  may change the complex structure). The fluctuation  $a$  in (2.15) is then a linear combination

$$a = D_{\xi^k}\mathcal{A}d\xi^k + D_{g^i}\mathcal{A}dg^i \quad (2.17)$$

of the covariant derivatives of  $\mathcal{A}$ .

In order to compute the metric on the bundle moduli space  $\mathcal{M}_F(g)$ , one should dimensionally reduce the gauge kinetic term in  $D = 10$  heterotic supergravity, assuming that the gauge fields depend on non-compact directions  $x$  through the bundle and metric moduli,  $\xi^k(x)$  and  $g^i(x)$ . The fluctuation (2.17) then represents the part of the gauge field strength with one leg on  $\mathfrak{S}$  and one leg in the non-compact directions. This results in [14]

$$e^{\rho/2}ds_F^2 = e^\rho \mathcal{G}_{k\bar{\ell}}(d\xi^k + C_i^k dg^i)(d\bar{\xi}^{\bar{\ell}} + \bar{C}_j^{\bar{\ell}} dg^j), \quad (2.18)$$

where

$$\mathcal{G}_{k\bar{\ell}} = \int_{\mathfrak{S}} \text{Tr } D_{\xi^k}\mathcal{A} \wedge \star D_{\bar{\xi}^{\bar{\ell}}}\mathcal{A}^\dagger, \quad C_i^k = \mathcal{G}^{k\bar{\ell}} \int_{\mathfrak{S}} \text{Tr } D_{\bar{\xi}^{\bar{\ell}}}\mathcal{A}^\dagger \wedge \star D_{g^i}\mathcal{A}. \quad (2.19)$$

By dimensional analysis,  $\mathcal{G}_{k\bar{\ell}}$  scales with the square root of the volume of  $\mathfrak{S}$ , so that (2.18) scales as  $e^{\rho/2}$ , justifying the notation on the l.h.s. For fixed geometric moduli, the metric  $\mathcal{G}_{k\bar{\ell}}$  is Kähler, with Kähler potential  $\int_{\mathfrak{S}} \text{Tr } \mathcal{A} \wedge \star \mathcal{A}^\dagger$  [30].

The previous discussion only relied on the Hermitian Yang-Mills equations (2.13) in a fixed complex structure, and produced a Kähler metric on bundle moduli space  $\mathcal{M}_F(g)$  for any Calabi-Yau compactifications. For compactifications on a  $K3$  surface  $\mathfrak{S}$ , the bundle moduli space  $\mathcal{M}_F(g)$  carries additional structure. First, as already mentioned, the equivalence with (2.13) holds for any complex structure on  $\mathfrak{S}$ , and produces a  $S^2$  worth of complex structures on  $\mathcal{M}_F(g)$ . In addition,  $\mathcal{M}_F(g)$  inherits a holomorphic symplectic structure from the holomorphic symplectic structure on  $\mathfrak{S}$  [31]. The holomorphic symplectic form on  $\mathcal{M}_F$  is given by the natural inner product on ( $\text{End}F$ -valued) one-forms,

$$\langle \beta_1, \beta_2 \rangle = \int_{\mathfrak{S}} \text{Tr } \beta_1 \wedge \star \beta_2. \quad (2.20)$$

Along with the Kähler form, the holomorphic symplectic form (2.20) provides a hyperkähler structure on  $\mathcal{M}_F(g)$ .



In addition, when  $\mathfrak{S}$  is an elliptic fibration  $\mathcal{E} \rightarrow \mathfrak{S} \rightarrow \mathcal{B}$ ,  $\mathcal{M}_F(g)$  is in fact a complex integrable system [22]. This can be seen from spectral cover construction of bundles on elliptic fibrations [32, 33] (see e.g. [34] for a non-technical discussion). The restriction of the  $SU(N)$  bundle on the elliptic fibers produces an  $N : 1$  covering  $\mathcal{C}$  of the base  $\mathcal{B}$  known as the spectral (or cameral) curve. Using the isomorphism between  $\mathcal{E}$  and its Jacobian,  $\mathcal{C}$  can be viewed as an effective curve inside  $\mathfrak{S}$  homologous to  $k\mathcal{E} + N\mathcal{B}$ , where  $k = c_2(F)$  [22]. To recover the full bundle on  $\mathfrak{S}$ , it is necessary to specify a degree  $g - 1 - N$  line bundle on  $\mathcal{C}$ , where  $g = N(k - N) + 1$  is the genus of the curve  $\mathcal{C}$  [35]. The choice of spectral cover is parametrized by a complex projective space  $\mathbb{P}^g$  (the linear system associated to the divisor  $k\mathcal{E} + N\mathcal{B}$ ), while the choice of line bundle is parametrized by the Jacobian torus of  $\mathcal{C}$ , of complex dimension  $g$ . This realizes the complex integrable system mentioned earlier. In practice, one can decompose the complex coordinates  $\xi^k$  into (complex) action and angle variables  $(t^\alpha, w_\alpha)$  such that the holomorphic symplectic form is  $dt^\alpha \wedge dw_\alpha$ , while the metric (at fixed metric moduli) becomes

$$e^{\rho/2} ds_F^2 = e^{\frac{1}{2}(\rho-R)} \gamma_{\alpha\beta} dt^\alpha d\bar{t}^\beta + e^{\frac{1}{2}(\rho+R)} \tilde{\gamma}^{\alpha\beta} (dw_\alpha + \mathcal{W}_\alpha)(d\bar{w}_\beta + \bar{\mathcal{W}}_\beta). \quad (2.21)$$

The exponential factor in front of the first term corresponds to the inverse volume of the elliptic fiber  $\mathcal{E}$ . In the limit  $R \rightarrow -\infty$ , the volume of the base  $\mathcal{B}$  is given by  $e^{-\frac{1}{2}(\rho+R)}$  (the inverse of the exponential factor in front of the second term) and is much larger than the fiber volume. When this happens, the metrics  $\gamma_{\alpha\beta}$ ,  $\tilde{\gamma}^{\alpha\beta}$  and the connection  $\mathcal{W}_\alpha$  are expected to depend only on the spectral curve moduli  $t^\alpha$ , so that the metric is flat along the torus fibers. In this case, both  $\tilde{\gamma}^{\alpha\beta}$  and  $\mathcal{W}_\alpha$  are fixed from the Kähler metric  $\gamma_{\alpha\beta}$  by requiring the total space to be hyperkähler (see [23] for a construction of the semi-flat hyperkähler metric on the cotangent bundle of a Kähler manifold).

### 2.3 Kalb-Ramond moduli

Finally, we consider the  $B$ -field moduli. For given fixed metric and gauge bundle, they parametrize solutions of the field equations

$$d \star_g dB + \frac{1}{4} d \star_g (\omega_G - \omega_L) = 0. \quad (2.22)$$

To compute the metric on the  $B$ -moduli space, it is convenient to first dualize the ten-dimensional Kalb-Ramond two-form  $B$  into a 6-form potential  $C_6$ , which leads to the following action

$$S_{10} = \int_{\mathbb{R}^6 \times \mathfrak{S}} \left[ -\frac{1}{2} dC_6 \wedge \star dC_6 + \frac{1}{4} dC_6 \wedge (\omega_G - \omega_L) \right]. \quad (2.23)$$

The Kaluza-Klein reduction on  $\mathfrak{S}$  proceeds by decomposing  $C_6 = \omega_I \wedge C_4^I$  where  $C_4^I$  are 4-forms in the six non-compact dimensions (indices are raised and lowered using  $\eta_{IJ}$ ). In the six-dimensional Einstein frame, the resulting action reads

$$S_6 = \int_{\mathbb{R}^6} \left[ -\frac{1}{2} e^{-\rho} M_{IJ} dC_4^I \wedge \star dC_4^J + \frac{1}{4} dC_4^I \wedge \int_{\mathfrak{S}} \omega_I \wedge (\omega_G - \omega_L) \right], \quad (2.24)$$

where we used (2.3). The integral in the last term turns the 5-form  $\omega_I \wedge (\omega_G - \omega_L)$  on  $\mathbb{R}^6 \times \mathfrak{S}$  into a one-form  $\mathcal{V}_I$  on  $\mathbb{R}^6$ . Dualizing the 4-forms  $C_4^I$  into compact scalars  $b_I$  in 6 dimensions leads finally to a metric

$$e^\rho ds_B^2 = e^\rho M^{IJ} (db_I + \mathcal{V}_I)(db_J + \mathcal{V}_J). \quad (2.25)$$

It describes a torus bundle of rank  $n + 2$ , with constant metric along the torus action but non-trivial curvature over the space of metric and bundle moduli. Denoting this curvature by  $c_1(b_I)$ , one finds

$$c_1(b_I) = d\mathcal{V}_I = \frac{1}{4} \int_{\mathfrak{S}} \omega_I \wedge (\text{Tr} F \wedge F - \text{Tr} R \wedge R). \quad (2.26)$$

The fact that the coordinates  $b^I$  are compact with period one requires that the curvatures  $c_1(b_I)$  should have integer periods on any non-contractible two-cycle inside the moduli space  $\mathcal{M}_{g,F}$  over which the torus  $T^{n+2}$  parametrized by  $b^I$  is fibered.

This description was used in [4] to argue that the hypermultiplet moduli space near an  $A_1$  singularity, in the limit where gravity decouples, is the moduli space of  $SU(2)$ , charge one instantons, also known as the Atiyah-Hitchin manifold. The flatness of the metric along  $T^{n+2}$  holds only in the large volume limit, since worldsheet instantons wrapping genus zero curves in  $\mathfrak{S}$  depend non-trivially on the  $B$ -moduli. The metric  $M_{IJ}$  may as well receive perturbative  $\alpha'$  corrections at finite volume. The connection  $\mathcal{V}_I$ , however, is fixed by the Chern-Simons coupling in 10 dimensions and related to the chiral anomaly on the heterotic worldsheet [36], so is expected to be exact.

Let us now extract the component of the curvature (2.26) along the bundle moduli space  $\mathcal{M}_F(g)$ . The relevant part of the field strength  $F$  that contributes to (2.26) is  $\partial_x \mathcal{A} + \text{c.c.}$ , where the derivative is along the non-compact directions, leading to

$$c_1(b_I)|_{\mathcal{M}_F(g)} = -2\mathcal{M}_{I,\bar{k}\ell} d\bar{\xi}^{\bar{k}} \wedge d\xi^\ell - (\mathcal{N}_{I,k\ell} d\xi^k \wedge d\xi^\ell + \text{c.c.}), \quad (2.27)$$

where

$$\mathcal{M}_{I,\bar{k}\ell} = \frac{1}{4} \int_{\mathfrak{S}} \omega_I \wedge \text{Tr} D_{\bar{\xi}^{\bar{k}}} \mathcal{A}^\dagger \wedge D_{\xi^\ell} \mathcal{A}, \quad \mathcal{N}_{I,k\ell} = \frac{1}{4} \int_{\mathfrak{S}} \omega_I \wedge \text{Tr} D_{\xi^k} \mathcal{A} \wedge D_{\xi^\ell} \mathcal{A}. \quad (2.28)$$

Notice that  $\text{Tr} D_{\xi^k} \mathcal{A} \wedge D_{\xi^\ell} \mathcal{A}$  and  $\text{Tr} D_{\bar{\xi}^{\bar{k}}} \mathcal{A}^\dagger \wedge D_{\xi^\ell} \mathcal{A}$  are well defined standard (closed) two-forms (i.e. they are singlets with respect to the gauge group), respectively of degree  $(0, 2)$  and  $(1, 1)$ . The former will then be proportional to the only  $(0, 2)$  form, i.e. the complex conjugate of the holomorphic two-form  $\mathcal{J}$ , while the latter will have a component along the Kähler form  $\mathcal{J}^3$  and one piece orthogonal to it. In particular

$$\begin{aligned} \text{Tr} D_{\bar{\xi}^{\bar{k}}} \mathcal{A}^\dagger \wedge D_{\xi^\ell} \mathcal{A} &= \left( \frac{\int_{\mathfrak{S}} \mathcal{J}^3 \wedge \text{Tr} D_{\bar{\xi}^{\bar{k}}} \mathcal{A}^\dagger \wedge D_{\xi^\ell} \mathcal{A}}{\int_{\mathfrak{S}} \mathcal{J}^3 \wedge \mathcal{J}^3} \right) \mathcal{J}^3 + \text{Tr} D_{\bar{\xi}^{\bar{k}}} \mathcal{A}^\dagger \wedge D_{\xi^\ell} \mathcal{A}|_{\perp} \\ &= \frac{i}{2} e^\rho \mathcal{G}_{\bar{k}\ell} \mathcal{J}^3 + \text{Tr} D_{\bar{\xi}^{\bar{k}}} \mathcal{A}^\dagger \wedge D_{\xi^\ell} \mathcal{A}|_{\perp}, \end{aligned} \quad (2.29)$$

where  $\text{Tr } D_{\bar{\xi}^k} \mathcal{A}^\dagger \wedge D_{\xi^\ell} \mathcal{A}|_\perp$  is the component orthogonal to  $\mathcal{J}^3$ . In the second line we have used the fact that Hodge duality on a one-form  $\alpha$  on  $\mathfrak{S}$  acts as  $\star \alpha = -i \mathcal{J}^3 \wedge \alpha$ , so that

$$\int_{\mathfrak{S}} \mathcal{J}^3 \wedge \text{Tr } D_{\bar{\xi}^k} \mathcal{A}^\dagger \wedge D_{\xi^\ell} \mathcal{A} = i \int_{\mathfrak{S}} \text{Tr } (\star D_{\bar{\xi}^k} \mathcal{A}^\dagger) \wedge D_{\xi^\ell} \mathcal{A} = i \mathcal{G}_{\bar{k}\ell}. \quad (2.30)$$

These relations allow to write the coefficients  $\mathcal{M}_{I,\bar{k}\ell}$  and  $\mathcal{N}_{I,k\ell}$  as

$$\mathcal{M}_{I,\bar{k}\ell} = \frac{i}{8} e^\rho \mathcal{G}_{\bar{k}\ell} \int_{\mathfrak{S}} \omega_I \wedge \mathcal{J}^3 + \widetilde{\mathcal{M}}_{I,\bar{k}\ell} \quad \text{with} \quad \widetilde{\mathcal{M}}_{I,\bar{k}\ell} \gamma_x^I = 0, \quad (2.31)$$

$$\mathcal{N}_{I,k\ell} = \frac{1}{4} e_{k\ell} \int_{\mathfrak{S}} \omega_I \wedge \bar{\mathcal{J}} = \frac{1}{4} e^{-\rho/2} \bar{\gamma}_I e_{k\ell}. \quad (2.32)$$

In particular, the results (2.31) imply that the curvatures  $\mathcal{M}_{\bar{k}\ell}^I$  satisfy

$$\gamma_I \mathcal{M}_{\bar{k}\ell}^I = 0, \quad \gamma_I^3 \mathcal{M}_{\bar{k}\ell}^I = \frac{i}{4} e^{\rho/2} \mathcal{G}_{\bar{k}\ell}. \quad (2.33)$$

## 2.4 Two-stage fibration and standard embedding locus

The upshot of this discussion is that in the limit where the volume  $\mathcal{V} = e^{-\rho}$  of  $\mathfrak{S}$  is large, the metric on the hypermultiplet moduli space  $\mathcal{M}_H$  takes the form

$$ds^2 = ds_g^2 + e^{\rho/2} ds_F^2 + e^\rho ds_B^2 \quad (2.34)$$

exhibiting a two-stage fibration structure

$$\begin{array}{ccc} T^{n+2} & \longrightarrow & \mathcal{M}_H \\ & & \downarrow \\ \mathcal{M}_F(g) & \longrightarrow & \mathcal{M}_{g,F} \\ & & \downarrow \\ & & \mathbb{R}^+ \times \frac{SO(3,n-1)}{SO(3) \times SO(n-1)} \end{array} \quad (2.35)$$

Here,  $\mathcal{M}_F(g)$  is a hyperkähler space parametrizing anti-self dual connections (of fixed topological type) on  $\mathfrak{S}$  with fixed metric  $g$ , of quaternionic dimension  $m$  given in (2.12), while the torus  $T^{n+2}$  parametrizes the  $B$ -field. The metric along the bundle moduli  $\mathcal{M}_F(g)$  scales like  $e^{\rho/2}$  and shrinks in the large volume limit  $\rho \rightarrow -\infty$ , while the metric along the  $B$ -field moduli scales like  $e^\rho$  and is even smaller. When  $\mathfrak{S}$  is elliptically fibered, with a base much larger than the fiber, the moduli space  $\mathcal{M}_F(g)$  acquires the structure of a complex integrable model, with a semi-flat hyperkähler metric. The torus  $T^{n+2}$  is non-trivially fibered over the moduli space  $\mathcal{M}_{g,F}$  of metrics and bundles, with curvature given by (2.26), while the bundle moduli space  $\mathcal{M}_F(g)$  is non-trivially fibered over the metric moduli space.

It is important to note that the metric (2.34) was obtained by reducing tree-level supergravity in 10 dimensions, which is not by itself supersymmetric. Thus, it is not expected to be QK. To obtain a metric consistent with supersymmetry, one should perform the Kaluza-Klein reduction of the fully supersymmetric,  $R^2$ -corrected supergravity action [5], a daunting task that we leave for future work.

A special class of anti-self dual connections is provided by deformations of the spin connection for the HK metric on  $\mathfrak{S}$ . This leads to bundles with  $c_2(F)=24$  and structure group  $SU(2)$ , whose commutant inside  $E_8 \times E_8$  is  $E_7 \times E_8$ . The bundle moduli space has dimension  $m = 45$ , leading to a 65-dimensional hypermultiplet space  $\mathcal{M}_H$  of the type above, with  $n = 20$ . Inside this space, there exists a submanifold corresponding to the locus where the gauge connection is equal (up to gauge transformation) to the spin connection. Since the worldsheet SCFT has enhanced  $(4, 4)$  supersymmetry at this point, its moduli space is entirely determined to be the symmetric space  $SO(4, 20)/SO(4) \times SO(20)$  [37]. This requires a delicate cancellation between the connection terms appearing in the metric on  $\mathcal{M}_F(g)$ , which will remain after freezing the bundle moduli, against perturbative corrections in the sigma model.

### 3 Two-stage fibration from heterotic/type II duality

In this section, we use heterotic/type II duality to get insight into the structure of the two-stage fibration (2.35), in particular into the fibration of the Kalb-Ramond torus  $T^{n+2}$  over the bundle and metric moduli. The key idea is that the large volume limit  $\rho \rightarrow -\infty$  on the heterotic side (combined with  $R \rightarrow -\infty$  with  $|R| \ll |\rho|$ ) corresponds to weak coupling on the type IIB side. In this limit, the hypermultiplet metric is obtained by the  $c$ -map procedure [38] from the moduli space of Kähler structure deformations. Our aim is to express the  $c$ -map metric in terms of heterotic variables, expose the fibration structure and read off the corresponding connections.

#### 3.1 Heterotic/type II duality in hypermultiplet sector

We first recall the basic features of heterotic/type II duality in the hypermultiplet sector [9–11, 19, 20].<sup>4</sup> According to this duality,  $E_8 \times E_8$  heterotic string theory compactified on  $\mathfrak{S} \times T^2$  is equivalent to type IIA string theory compactified on a Calabi-Yau threefold  $\mathfrak{Y}$ , or to type IIB string theory compactified on the mirror CY threefold  $\hat{\mathfrak{Y}}$ . The topology of  $\mathfrak{Y}$  and  $\hat{\mathfrak{Y}}$  depends on the topology of the gauge bundle on the heterotic side, but a general fact is that both  $\mathfrak{Y}$  and  $\hat{\mathfrak{Y}}$  must admit a  $K3$  fibration [19, 21, 39]. We shall focus on the type IIB description and denote by  $\Sigma$  the fiber in the  $K3$  fibration  $\Sigma \rightarrow \hat{\mathfrak{Y}} \rightarrow \mathbb{P}$ .

In general, the HM moduli space in type IIB string theory compactified on  $\hat{\mathfrak{Y}}$  is parametrized by the four-dimensional string coupling  $g_{(4)} \equiv 1/\sqrt{r}$ , the Kähler moduli  $z^a$ , the periods  $\zeta^\Lambda, \tilde{\zeta}_\Lambda$  of the Ramond-Ramond potentials on  $H_{\text{even}}(\hat{\mathfrak{Y}}, \mathbb{Z})$  and the Neveu-Schwarz axion  $\sigma$ , dual to the Kalb-Ramond two-form in 4 dimensions. The duality identifies [19, 20]

$$r = \frac{1}{2} e^{-\frac{1}{2}(\rho+R)}, \quad \text{Re } s = e^{-\frac{1}{2}(\rho-R)}, \quad (3.1)$$

where  $\text{Re } s$  is the area of the base  $\mathbb{P}$  of the  $K3$  fibration, so the large volume limit  $\rho \rightarrow -\infty$  on the heterotic side corresponds to  $g_{(4)} \rightarrow 0$  and  $\text{Re } s \rightarrow +\infty$  on the type IIB side. The

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<sup>4</sup>Since the hypermultiplet moduli space is independent of the  $T^2$  moduli, one may as well work in 6 dimensions, and use heterotic/F-theory duality, where F-theory is compactified on the same CY threefold  $\hat{\mathfrak{Y}}$ .

ten-dimensional string coupling is however finite in this limit [20],

$$\frac{1}{g_s} = 4\sqrt{r} e^{\mathcal{K}/2} = \frac{|X^0| e^{-R/2}}{\sqrt{\eta^{AB} X_A \bar{X}_B}}. \quad (3.2)$$

Thus, in order for all quantum corrections on the type IIB side to be exponentially suppressed, one should further take  $R \rightarrow -\infty$ , with  $|R| \ll |\rho|$  so that  $\text{Re } s$  remains very large.

In this limit, the hypermultiplet moduli space on the type IIB side is obtained by the (local)  $c$ -map procedure from the Kähler moduli space of  $\hat{\mathfrak{Y}}$  [38, 40]. In the limit  $\text{Re } s \rightarrow +\infty$ , the prepotential takes the form

$$F(Z^\Lambda) = -\frac{Z^s \eta_{ij} Z^i Z^j}{2Z^0} + f(Z^0, Z^i, Z^\alpha) + \mathcal{O}(e^{2\pi i Z^s/Z^0}), \quad (3.3)$$

where  $Z^s/Z^0 = is$  is the complex Kähler modulus associated to the base while  $Z^i/Z^0 = it^i$  are complex Kähler moduli for two-cycles  $\gamma^i$  in the  $K3$  fiber  $\Sigma$ , with  $\eta_{ij}$  being their intersection matrix. The remaining variables  $Z^\alpha/Z^0 = it^\alpha$  are complex Kähler moduli for the remaining two-cycles  $\gamma^\alpha$  in  $H^2(\hat{\mathfrak{Y}}, \mathbb{Z})$ , dual to reducible singular fibers of the  $K3$  fibration [19]. These singular fibers do not intersect the section, hence the corresponding Kähler moduli  $t^\alpha$  do not appear in the leading term in (3.3). In contrast, the next-to-leading order term  $f(Z^0, Z^i, Z^\alpha) = (Z^0)^2 f(1, it^i, it^\alpha)$  does depend on all Kähler moduli except  $s$ , and includes effects of all worldsheet instantons wrapping the two-cycles  $\gamma^i, \gamma^\alpha$ , in addition to the classical cubic contribution, because the Kähler moduli  $t^i, t^\alpha$  stay finite in the large volume limit  $\rho \rightarrow -\infty$  on the heterotic side.

In [19, 20], it was shown that in the absence of reducible singular fibers, the QK metric derived from the leading term (3.3) describes the symmetric space  $SO(4, n)/SO(4) \times SO(n)$ , reproducing the expected hypermultiplet moduli space for heterotic strings compactified on  $K3$  equipped with a rigid gauge bundle. Under this identification, the heterotic and type II variables were related by (3.1) supplemented by

$$\text{Re}(is) = B_2, \quad c^A = v^A, \quad \tilde{c}_A = B_A - B_2 v_A, \quad \sigma = -2B_1 - v^A B_A, \quad (3.4)$$

where  $c^A, \tilde{c}_A$  are related to  $\zeta^A, \tilde{\zeta}_A$  by a symplectic rotation  $(Z^s, F_s) \mapsto (F_s, -Z^s)$ , whereas the moduli  $t^i$  are related to the complex structure moduli  $X^A$  on the heterotic side via (3.10) below. Here the index  $A$  runs over the  $n$  values  $(0, s, i)$  so that  $\Lambda = (A, \alpha)$ . The additional moduli  $t^\alpha$  and  $c^\alpha, \tilde{c}_\alpha$  associated with reducible bad fibers can be identified with bundle moduli on the heterotic side [9, 11]. The leading contribution to the metric along these directions comes from the second term in (3.3), which can no longer be ignored. As we shall now demonstrate, the QK space obtained by the  $c$ -map procedure from the prepotential (3.3), keeping only the first two terms<sup>5</sup>, has the two-stage fibration structure (2.35). Furthermore, the metric on the HK fiber  $\mathcal{M}_F(g)$  is the rigid  $c$ -map space [40] derived from the prepotential  $f(Z^0, Z^i, Z^\alpha)$  for fixed  $Z^0, Z^i$ .

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<sup>5</sup>All other corrections to the prepotential correspond to the worldsheet instantons wrapped on the base  $\mathbb{P}$  of the  $K3$  fibration and are exponentially suppressed in our scaling limit. Thus,  $f$  is the only relevant correction.

### 3.2 $c$ -map in heterotic variables

The QK metric obtained by the  $c$ -map procedure from a special Kähler manifold with holomorphic symplectic section  $\Omega(z^a) = (Z^\Lambda, F_\Lambda)$  takes the form

$$\begin{aligned} ds^2 = & \frac{1}{r^2} dr^2 - \frac{1}{2r} (\text{Im } \mathcal{N})^{\Lambda\Sigma} \left( d\tilde{\zeta}_\Lambda - \mathcal{N}_{\Lambda\Lambda'} d\zeta^{\Lambda'} \right) \left( d\tilde{\zeta}_\Sigma - \bar{\mathcal{N}}_{\Sigma\Sigma'} d\zeta^{\Sigma'} \right) \\ & + \frac{1}{16r^2} \left( d\sigma + \tilde{\zeta}_\Lambda d\zeta^\Lambda - \zeta^\Lambda d\tilde{\zeta}_\Lambda \right)^2 + 4\mathcal{K}_{a\bar{b}} dz^a d\bar{z}^{\bar{b}}, \end{aligned} \quad (3.5)$$

where  $\mathcal{K}_{a\bar{b}}$  is the Kähler metric on the special Kähler manifold,

$$\mathcal{K}_{a\bar{b}} = \partial_{z^a} \partial_{\bar{z}^{\bar{b}}} \mathcal{K}, \quad \mathcal{K} = -\log \left[ i \left( \bar{Z}^\Lambda F_\Lambda - Z^\Lambda \bar{F}_\Lambda \right) \right], \quad (3.6)$$

and  $\mathcal{N}_{\Lambda\Sigma}$  is a symmetric complex matrix with negative definite imaginary part, determined by the conditions [41, 42]

$$F_\Lambda = \mathcal{N}_{\Lambda\Sigma} Z^\Sigma, \quad D_a F_\Lambda = \bar{\mathcal{N}}_{\Lambda\Sigma} D_a Z^\Sigma, \quad (3.7)$$

where  $D_a = \partial_a + \partial_a \mathcal{K}$  is the Kähler covariant derivative. When  $\Omega(z^a)$  derives from a homogeneous prepotential  $F(Z^\Lambda)$ , i.e.  $F_\Lambda \equiv \partial_{Z^\Lambda} F$ , the matrix  $\mathcal{N}_{\Lambda\Sigma}$  is given in terms of the second derivative  $F_{\Lambda\Sigma} = \partial_{Z^\Lambda} \partial_{Z^\Sigma} F$  via

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i \frac{\text{Im } F_{\Lambda\Lambda'} Z^{\Lambda'} \text{Im } F_{\Sigma\Sigma'} Z^{\Sigma'}}{Z^\Xi \text{Im } F_{\Xi\Xi'} Z^{\Xi'}}. \quad (3.8)$$

However, for the purposes of heterotic/type II duality, it is more convenient to work in a different symplectic basis where the prepotential does not exist [41, 43]. The section  $(X^\Lambda, G_\Lambda)$  in this new basis is related to the section  $(Z^\Lambda, F_\Lambda)$  in which (3.3) applies by the symplectic transformation on the symplectic plane associated to the Kähler modulus  $s$

$$X^s = F_s, \quad G_s = -Z^s. \quad (3.9)$$

We denote by  $c^\Lambda, \tilde{\zeta}_\Lambda$  the coordinates  $\zeta^\Lambda, \tilde{\zeta}_\Lambda$  in this new basis. The new symplectic section is given by

$$X^\Lambda = X^0 (1, \frac{1}{2} \eta_{ij} t^i t^j, it^i, it^\alpha), \quad G_\Lambda = -is X_\Lambda + f_\Lambda, \quad (3.10)$$

where  $f_\Lambda = (\partial_{Z^0}, 0, \partial_{Z^i}, \partial_{Z^\alpha}) f$  and  $X_\Lambda = \eta_{\Lambda\Sigma} X^\Sigma$  with  $\eta_{\Lambda\Sigma}$  the degenerate symmetric matrix

$$\eta_{\Lambda\Sigma} = \begin{pmatrix} \eta_{AB} & 0 \\ 0 & 0 \end{pmatrix}, \quad \eta_{AB} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \eta_{ij} \end{pmatrix}. \quad (3.11)$$

Note that  $X^A$  satisfy the same constraint  $X^A \eta_{AB} X^B = 0$  as in section 2.1 upon identification of  $\eta_{AB}$  with the intersection form of 2-cycles on  $K3$ .

The Kähler potential and period matrix read

$$\begin{aligned} \mathcal{K} = & -\log (\text{Re } s - \mathfrak{f}) - \log |t + \bar{t}|^2 - \log |X^0|^2, \\ \mathcal{N}_{\Lambda\Sigma} = & i\bar{s} \eta_{\Lambda\Sigma} - i(s + \bar{s}) X_{\Lambda\Sigma} + \bar{f}_{\Lambda\Sigma} \\ & + 2i \frac{\bar{X}_\Lambda X^\Xi \text{Im } f_{\Sigma\Xi} + \bar{X}_\Sigma X^\Xi \text{Im } f_{\Lambda\Xi}}{\eta_{CD} X^C \bar{X}^D} - 2i \frac{\bar{X}_\Lambda \bar{X}_\Sigma}{(\eta_{CD} X^C \bar{X}^D)^2} X^\Xi X^\Theta \text{Im } f_{\Xi\Theta}, \end{aligned} \quad (3.12)$$

where  $|t + \bar{t}|^2 = \eta_{ij}(t^i + \bar{t}^i)(t^j + \bar{t}^j)$  and we introduced

$$\mathfrak{f} = \frac{1}{2} X^{\Lambda\Sigma} \text{Im} f_{\Lambda\Sigma}, \quad X^{\Lambda\Sigma} = \frac{X^\Lambda \bar{X}^\Sigma + \bar{X}^\Lambda X^\Sigma}{\eta_{CD} X^C \bar{X}^D}, \quad (3.13)$$

while indices on  $X^{\Lambda\Sigma}$  are lowered using (3.11). The function  $\mathfrak{f}$ , a real function of  $t^i, t^\alpha$ , will play an important role below. It determines the order  $\mathcal{O}(1/\text{Re } s)$  correction to the Kähler potential  $\mathcal{K}$  in the limit  $\text{Re } s \rightarrow +\infty$ . The matrix  $X^{\Lambda\Sigma}$  generalizes  $X^{AB}$  in (2.8). Splitting the index  $\Lambda$  into  $A, \alpha$ , the inverse of the imaginary part of the matrix  $\mathcal{N}_{\Lambda\Sigma}$  can be computed to be

$$\begin{aligned} \text{Im} \mathcal{N}^{AB} &= (\delta_C^A - X^A_C) V^{CD} (\delta_D^B - X_D^B) - \frac{X^{AB}}{\text{Re } s - \mathfrak{f}}, \\ \text{Im} \mathcal{N}^{A\alpha} &= -(\delta_C^A - X^A_C) V^{CD} (\delta_D^B - X_D^B) \nu_{B\beta} \mu^{\alpha\beta} - \frac{X^{A\alpha}}{\text{Re } s - \mathfrak{f}}, \\ \text{Im} \mathcal{N}^{\alpha\beta} &= -\mu^{\alpha\beta} + \mu^{\alpha\gamma} \nu_{A\gamma} (\delta_C^A - X^A_C) V^{CD} (\delta_D^B - X_D^B) \nu_{B\delta} \mu^{\delta\beta} - \frac{X^{\alpha\beta}}{\text{Re } s - \mathfrak{f}}, \end{aligned} \quad (3.14)$$

where we introduced the following notations:

$$\begin{aligned} \mu_{\alpha\beta} &= \text{Im} f_{\alpha\beta}, \quad \nu_{A\alpha} = \text{Im} f_{A\alpha}, \quad \lambda_{AB} = \text{Im} f_{AB}, \\ V_{AB} &= \text{Re } s \eta_{AB} - (\delta_A^C - X_A^C) (\lambda_{CD} - \nu_{C\alpha} \mu^{\alpha\beta} \nu_{D\beta}) (\delta_B^D - X_B^D), \end{aligned} \quad (3.15)$$

and  $\mu^{\alpha\beta}, V^{AB}$  are the inverse of  $\mu_{\alpha\beta}$  and  $V_{AB}$ , respectively.

We now apply the same duality map (3.4) and (3.1), except for a minor change in the definition of the coordinates  $\rho, R$  involving the function  $\mathfrak{f}$ ,

$$r = \frac{1}{2} e^{-\frac{1}{2}(\rho+R)}, \quad \text{Re } s = e^{-\frac{1}{2}(\rho-R)} + \mathfrak{f}, \quad (3.16)$$

$$\text{Re}(is) = B_2, \quad c^A = v^A, \quad \tilde{c}_A = B_A - B_2 v_A, \quad \sigma = -2B_1 - v^A B_A. \quad (3.17)$$

Guided by similar definitions in the absence of bundle moduli  $t^\alpha$  [19, 20], we further define the symmetric matrix

$$\mathcal{M}_{AB} = M_{AB} + e^{\frac{1}{2}(\rho-R)} (\delta_A^C - X_A^C) (\lambda_{CD} - \nu_{C\alpha} \mu^{\alpha\beta} \nu_{D\beta} - \mathfrak{f} \eta_{CD}) (\delta_B^D - X_B^D) \quad (3.18)$$

with  $M_{AB}$  from (2.8), in such a way that its inverse  $\mathcal{M}^{AB}$  satisfies  $\mathcal{M}^{AB} = -e^{\frac{1}{2}(R-\rho)} \text{Im} \mathcal{N}^{AB}$ .  $\mathcal{M}_{AB}$  reduces to  $M_{AB}$  in the limit  $\rho \rightarrow -\infty$ , but in general is *not* an element of  $SO(2, n-2)$ . We also introduce a symmetric matrix

$$\mathcal{M}^{IJ} = \begin{pmatrix} e^R & -\frac{v^2}{2} e^R & e^R v^B \\ -\frac{v^2}{2} e^R & \frac{v^4}{4} e^R + e^{-R} + \mathcal{M}^{AB} v_A v_B & -\frac{v^2}{2} e^R v^B - \mathcal{M}^{BC} v_C \\ e^R v^A & -\frac{v^2}{2} e^R v^A - \mathcal{M}^{AC} v_C & e^R v^A v^B + \mathcal{M}^{AB} \end{pmatrix} \quad (3.19)$$

which reduces to the inverse of the matrix  $M^{IJ}$  from (2.3) in the limit  $\rho \rightarrow -\infty$ , but is in general not an element of  $SO(3, n-1)$ , except when  $\mathcal{M}^{AB}$  is an element of  $SO(2, n-2)$ .



With these definitions, the  $c$ -map metric (3.5) associated to the prepotential (3.3) (retaining only the first two terms, and assuming no special property of  $f(Z^0, Z^i, Z^\alpha)$  other than independence on  $Z^s$  and homogeneity) can be written as a sum of three contributions,

$$ds^2 = ds_g^2 + e^{\rho/2} ds_F^2 + e^\rho ds_B^2, \quad (3.20)$$

matching the expected form (2.34) of the hypermultiplet metric moduli space on the heterotic side. In the following we discuss each contribution in turn.

### 3.2.1 Metric moduli

The first term

$$ds_g^2 = \frac{1}{2} d\rho^2 + \frac{1}{2} dR^2 - \frac{1}{4} dM_{AB} dM^{AB} + e^R \mathcal{M}_{AB} dv^A dv^B \quad (3.21)$$

generalizes the  $SO(3, n-1)$  invariant metric (2.9) on the moduli space  $\mathcal{M}_g$  of HK metrics on  $\mathfrak{S}$ . It reduces to this invariant metric in the large volume limit  $\rho \rightarrow -\infty$ , but contains in addition power-suppressed corrections due to the difference between  $\mathcal{M}_{AB}$  and  $M_{AB}$ .

### 3.2.2 Bundle moduli

The second term is given by

$$ds_F^2 = 4e^{-R/2} \partial \bar{\partial} \mathfrak{f} + e^{R/2} \mu_{\alpha\beta} Dc^\alpha Dc^\beta + e^{R/2} \mu^{\alpha\beta} (D\tilde{c}_\alpha - \text{Re } f_{\alpha\alpha'} dc^{\alpha'}) (D\tilde{c}_\beta - \text{Re } f_{\beta\beta'} dc^{\beta'}), \quad (3.22)$$

where

$$\partial \bar{\partial} \mathfrak{f} = \frac{\mu_{\alpha\beta} DX^\alpha D\bar{X}^\beta}{\eta_{CD} X^C \bar{X}^D} + (\delta_A^C - X_A^C) (\lambda_{CD} - \nu_{C\alpha} \mu^{\alpha\beta} \nu_{D\beta} - \mathfrak{f} \eta_{CD}) (\delta_B^D - X_B^D) \frac{dX^A d\bar{X}^B}{\eta_{CD} X^C \bar{X}^D}. \quad (3.23)$$

It contains kinetic terms for the bundle moduli<sup>6</sup>  $t^\alpha, c^\alpha$  and  $\tilde{c}_\alpha$ , with specific connections with respect to the metric moduli,

$$DX^\alpha = dX^\alpha - \Gamma_A^\alpha dX^A, \quad Dc^\alpha = dc^\alpha - \Gamma_A^\alpha dv^A, \quad D\tilde{c}_\alpha = d\tilde{c}_\alpha - \text{Re } \mathcal{N}_{\alpha A} dv^A, \quad (3.24)$$

where

$$\Gamma_A^\alpha = \mu^{\alpha\beta} \text{Im } \mathcal{N}_{A\beta} = X_A^\alpha - (\delta_A^B - X_A^B) \nu_{B\beta} \mu^{\alpha\beta}. \quad (3.25)$$

Setting these connection terms to zero, i.e. fixing the HK metric on  $\mathfrak{S}$ , the metric (3.22) reduces to

$$ds_F^2|_g = 8e^{-R/2} \frac{\mu_{\alpha\beta} dt^\alpha d\bar{t}^\beta}{|t + \bar{t}|^2} + e^{R/2} \mu^{\alpha\beta} (dw_\alpha + \mathcal{W}_\alpha) (d\bar{w}_\beta + \bar{\mathcal{W}}_\beta), \quad (3.26)$$

where we denoted

$$w_\alpha = \tilde{c}_\alpha - f_{\alpha\beta} c^\beta, \quad \mathcal{W}_\alpha = -\frac{1}{2} \mu^{\gamma\lambda} (w_\lambda - \bar{w}_\lambda) f_{\alpha\beta\gamma} dt^\beta. \quad (3.27)$$

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<sup>6</sup>It also contains a contribution to the kinetic term for the metric moduli  $t^i$  generated by the second term in (3.23), which is however suppressed with respect to the leading contribution from  $ds_g^2$ .

We recognize in this expression the rigid  $c$ -map metric [40] associated to the prepotential  $f(Z^0, Z^i, Z^\alpha)$  for fixed  $Z^0, Z^i$  with  $(t^\alpha, w_\alpha)$  being holomorphic Darboux coordinates for the holomorphic symplectic form (2.20) on  $\mathcal{M}_F(g)$ . Eq. (3.26) is in agreement with the expected form of the metric (2.21) in coordinates adapted to the spectral cover construction. In particular, the real function  $\mathfrak{f}$  defined in (3.13) provides, up to an overall factor  $4e^{-R/2}$ , a Kähler potential for the Kähler metric on the moduli space  $\mathbb{P}^g$  of spectral covers mentioned above (2.21). It is somewhat unexpected that the metric (3.22) on bundle moduli space should belong to the class of rigid  $c$ -map metrics, rather than the more general class of semi-flat HK metrics on cotangent bundles of Kähler manifolds constructed in [23].

### 3.2.3 Kalb-Ramond moduli

Finally,

$$ds_B^2 = \mathcal{M}^{IJ} (dB_I + \mathcal{V}_I) (dB_J + \mathcal{V}_J) \quad (3.28)$$

reproduces the flat metric on the torus fiber  $T^{n+2}$  of the two-stage bundle (2.35). The kinetic term  $\mathcal{M}^{IJ}$  is in agreement with (2.25) in the large volume limit, while the connection reads<sup>7</sup>

$$\begin{aligned} \mathcal{V}_1 &= \frac{v^2}{2} \mathcal{V}_2 - v^A \mathcal{V}_A + \frac{1}{2} (c^\alpha d\tilde{c}_\alpha - \tilde{c}_\alpha dc^\alpha), \\ \mathcal{V}_2 &= -i(\partial - \bar{\partial})\mathfrak{f}, \\ \mathcal{V}_A &= -iv_A(\partial - \bar{\partial})\mathfrak{f} + \Gamma_A^\alpha d\tilde{c}_\alpha - (\text{Re}\mathcal{N}_{A\beta} + \Gamma_A^\alpha \text{Re}\mathcal{N}_{\alpha\beta}) dc^\beta \\ &\quad - (\text{Re}\mathcal{N}_{AB} + B_2 \eta_{AB} + \Gamma_A^\alpha \text{Re}\mathcal{N}_{\alpha B}) dv^B. \end{aligned} \quad (3.29)$$

We shall be particularly interested in the restriction of the curvatures  $d\mathcal{V}_I$  along the bundle moduli directions (i.e. for fixed metric  $g$  on  $\mathfrak{S}$ ), which can be decomposed into their (1,1) and (2,0) components,

$$d\mathcal{V}_I|_g = -2\mathcal{M}_{I,\bar{k}\ell} d\bar{\xi}^{\bar{k}} \wedge d\xi^\ell - (\mathcal{N}_{I,k\ell} d\xi^k \wedge d\xi^\ell + \text{c.c.}), \quad (3.30)$$

where  $\xi^k = (t^\alpha, w_\alpha)$  denote the holomorphic bundle moduli. Denoting also

$$\phi_{A\beta}^\alpha = \frac{1}{4} (\delta_A^B - X_A^B) \mu^{\alpha\gamma} (f_{\beta\gamma\lambda} \mu^{\lambda\delta} \text{Im} f_{B\delta} - f_{B\beta\gamma}), \quad (3.31)$$

$$\psi_\beta^\alpha = \frac{i}{8} f_{\beta\gamma\lambda} \mu^{\alpha\gamma} \mu^{\lambda\delta} (w_\delta - \bar{w}_\delta), \quad (3.32)$$

one finds

$$\begin{aligned} \mathcal{M}_{1,\bar{k}\ell} &= \begin{pmatrix} -\frac{iv^2\mu_{\alpha\beta}}{|t+\bar{t}|^2} + 4iv^A \mu_{\gamma\gamma'} \left( \bar{\phi}_{A\alpha}^\gamma \psi_\beta^{\gamma'} + \phi_{A\beta}^\gamma \bar{\psi}_\alpha^{\gamma'} \right) + 4i\mu_{\gamma\gamma'} \bar{\psi}_\alpha^\gamma \psi_\beta^{\gamma'} v^A \bar{\phi}_{A\alpha}^\beta + \bar{\psi}_\alpha^\beta & \\ -v^A \phi_{A\beta}^\alpha - \psi_\beta^\alpha & \frac{i}{4} \mu^{\alpha\beta} \end{pmatrix}, \\ \mathcal{M}_{2,\bar{k}\ell} &= \begin{pmatrix} \frac{2i\mu_{\alpha\beta}}{|t+\bar{t}|^2} & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{M}_{A,\bar{k}\ell} &= \begin{pmatrix} \frac{2iv_A\mu_{\alpha\beta}}{|t+\bar{t}|^2} - 4i\mu_{\gamma\gamma'} \left( \bar{\phi}_{A\alpha}^\gamma \psi_\beta^{\gamma'} + \phi_{A\beta}^\gamma \bar{\psi}_\alpha^{\gamma'} \right) - \bar{\phi}_{A\alpha}^\beta & \\ \phi_{A\beta}^\alpha & 0 \end{pmatrix} \end{aligned} \quad (3.33)$$

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<sup>7</sup>The term  $B_2 \eta_{AB}$  in the last term of  $\mathcal{V}_A$  cancels the dependence on the  $B$ -field introduced by  $\text{Re}\mathcal{N}_{AB}$ .

and

$$\mathcal{N}_{I,k\ell} = \frac{i\bar{\gamma}_I}{\sqrt{2}|t+\bar{t}|} \begin{pmatrix} 0 & -\delta_\alpha^\beta \\ \delta_\beta^\alpha & 0 \end{pmatrix}. \quad (3.34)$$

In particular, it is easily checked that  $\mathcal{N}_{I,k\ell}$  agrees with (2.32) given that  $t^\alpha$  and  $w_\alpha$  are Darboux coordinates for the symplectic matrix  $e_{k\ell}$ , whereas  $\mathcal{M}_{I,\bar{k}\ell}$  satisfies (2.33) with

$$\mathcal{G}_{\bar{k}\ell} = e^{\frac{1}{2}(R-\rho)} \begin{pmatrix} \frac{8e^{-R}\mu_{\alpha\beta}}{|t+\bar{t}|^2} + 16\mu_{\gamma\gamma'}\bar{\psi}_\alpha^\gamma\psi_\beta^{\gamma'} & -4i\bar{\psi}_\alpha^\beta \\ 4i\psi_\beta^\alpha & \mu^{\alpha\beta} \end{pmatrix} \quad (3.35)$$

being (up to an overall factor of  $e^{\rho/2}$ ) the metric on bundle moduli space read off from (3.26). Thus, heterotic/type II duality predicts the values of the integrals (2.28), in terms of the holomorphic prepotential  $f(t^i, t^\alpha)$  which governs the hyperkähler metric on bundle moduli space. It would be very interesting to compute the integrals (2.28) independently and extract the prepotential.

## 4 Discussion

In this work we have used heterotic/type II duality to shed light on the hypermultiplet moduli space in heterotic string theory compactified on an elliptically fibered  $K3$  surface, in the limit where the volume is very large and the base is much larger than the fiber. On the type IIA side, this corresponds to a limit where the string coupling vanishes while the size of the base of the  $K3$  fibration becomes infinite. In this limit, the classical cubic term in the prepotential (3.3) dominates the kinetic terms of the metric and B-field moduli, while the bundle moduli metric is determined by the subleading term in (3.3). The latter in general contains worldsheet instanton corrections wrapping two-cycles on the  $K3$  fiber and/or singular reducible fibers. By identifying the Kähler moduli  $t^\alpha$  and the corresponding Ramond-Ramond moduli  $c^\alpha, \tilde{c}_\alpha$  associated to these singular fibers with the bundle moduli  $\xi^k$  on the heterotic side, and using the slightly modified duality map of [19, 20] for the remaining moduli, we have found that the resulting moduli space has the two-stage fibration structure (2.35) expected on the heterotic side. In particular, the bundle moduli space  $\mathcal{M}_F(g)$  is a torus bundle over a Kähler space equipped with a semi-flat hyperkähler metric, as appropriate for a complex integrable system. The HK metric is obtained by the rigid  $c$ -map from the subleading term  $f(Z^0, Z^i, Z^\alpha)$  in the prepotential (3.3) (at fixed  $Z^0, Z^i$ ), a special case of the class of semi-flat HK metrics on cotangent bundles of Kähler manifolds constructed in [23]. It would be interesting to compare this prediction with a first principle computation of the metric on the moduli space of spectral covers.

In addition, the  $B$ -field moduli live in a torus  $T^{n+2}$  with flat metric along the torus action, and with non-trivial curvature over both the bundle moduli and the metric moduli. Remarkably, the component of the curvature along the bundle moduli are determined by the same holomorphic prepotential  $f(Z^0, Z^i, Z^\alpha)$  — a non-trivial prediction for the integrals appearing in (2.28). The metric on the torus  $T^{n+2}$ , given by the matrix  $\mathcal{M}^{IJ}$  (3.19), also receives corrections suppressed by inverse powers of the volume. These corrections are necessary for the metric

to be QK, and we expect that they could be derived by performing a careful Kaluza-Klein reduction of the fully supersymmetric,  $R^2$ -corrected heterotic supergravity in ten dimensions [5]. Before tackling this daunting task however, a basic problem is to construct a set of gauge-invariant coordinates on bundle and  $B$ -field moduli space. While the first issue requires a deeper understanding of the spectral cover construction, the second problem may be usefully circumvented by dualizing the Kalb-Ramond field to a six-form potential, as discussed in §2.3, and dualize back the resulting four-forms after reduction.

Finally, it is worth reiterating that the above difficulties arise just as well in the more phenomenologically appealing heterotic compactifications on Calabi-Yau threefolds. We hope that the insights gained through the present study can stimulate progress on  $\mathcal{N} = 1$  heterotic string vacua.

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